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Mass transfer from an axial source in a turbulent radial wall jet

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This paper considers the mixing of a turbulent radial wall jet with a secondary fluid introduced into the impingement area of the wall jet so as to form a steady axisymmetric state. Similarity of the concentration profiles perpendicular to the wall has been assumed and, by solving the momentum and mass flow equations, the concentration distribution through the layer and a similarity exponent giving the variation of concentration along the wall have been determined.

1. Introduction

The term 'wall jet' is used to describe the flow produced over a plate immersed in a fluid when a jet of the fluid impinges on the plate (usually at right angles). The region of interest is the fully developed flow that occurs after the stagnation and transition regions; if the impinging jet is at right angles, the wall jet spreads out radially and the fully developed flow lies at about four free-jet radii from the axis (Bradshaw & Love 1961).

The earliest work on the wall jet seems to have been done as part of an investigation of two-dimensional turbulent jet expansion by Förthmann (1934), under the heading of a 'partially open jet'. By making a non-dimensional plotting of velocity distribution across the wall jet against distance from the wall, y, he found the profiles to coincide in one curve thereby showing that the flow followed a simple law of similitude. He also observed that the boundary-layer thickness varied linearly with distance along the wall, x, the maximum velocity varied as $x^{-\frac{1}{2}}$ and the velocity in the region near to the wall varied as $y^{\frac{1}{7}}$.

The name 'wall jet' was first ascribed to this type of flow by Glauert (1956) in a theoretical investigation of laminar and turbulent flows for both two dimensional and radial wall jets. In the case of the turbulent wall jet, the eddy-viscosity varies over the flow and Glauert considered a hybrid structure in which the eddy-viscosity distribution near the wall is consistent with a modification of the power-law velocity profile due to Blasius (1913) for flow in a pipe (Schlichting 1955, p. 404), this being in keeping with Förthmann's findings, and the eddy-viscosity in the outer layer is found from Prandtl's hypothesis for free turbulent flow. Complete similarity is not now possible, since the eddy-viscosities in the inner and outer parts of the flow vary in slightly differing manners with the Reynold's number; however, Bakke (1957) found the variation to be so slow as to be undetectable by experiment. Since Bakke's experimental investigation, several authors (Bradshaw & Love 1961; Seban & Back 1961; Sigalla 1958; Bradshaw & Gee 1962) have verified experimentally the validity of the velocity distribution found by Glauert.

For an incompressible, turbulent radial wall jet Poreh & Tsuei (1965) deduce, by considering complete self-similarity of the velocity profiles, that the boundary-layer thickness varies as x and the maximum velocity as x^{-1}. However, they show that by neglecting viscous shear it is possible to have only approximate self-similarity with the maximum velocity varying as x^d. Experimental work by Tsuei (1962) confirms this approximation and evaluates d as $-1\cdot1$. Self-similarity of temperature profiles then implies that the maximum temperature Tm in a slightly heated jet impinging on an insulated wall varies as $x^{-(d+2)}$. Taking Tsuei's value of d gives $Tm \propto x^{-0\cdot9}$.

Laminar mixing between a radial wall jet of incompressible gas and a second gas was considered by Chaudhury (1964).

The present paper considers the mixing between a turbulent radial wall jet, produced by a jet of fluid impinging normally on a plate, and a secondary fluid emitted from the plate so as to produce an axisymmetric flow (see figure 1). Temperatures have been assumed constant and buoyancy effects have been neglected.

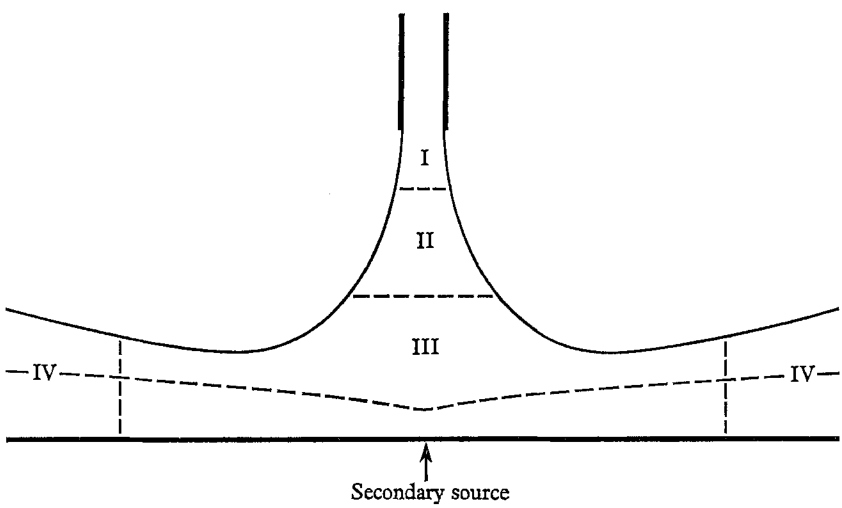

FIGURE 1. Flow diagram: I, transition zone from pipe flow to free-jet flow; II, free-jet zone; III, stagnation and transition zone; IV, wall-jet region.

A similarity solution of the concentration profile, in which the form of the concentration distribution through the jet does not vary with distance from the axis of symmetry has been assumed, and the same eddy-viscosity variation and velocity profile as formulated by Glauert have been used. Since this means that the inner part of the wall jet is governed by Blasius's formula and the outer part by Prandtl's hypothesis, the junction of the two regions being at the velocity maximum, some slight discontinuity in the concentration gradient at this point is to be found.

2. Theory

2·1. *Governing equations and boundary conditions*

Consider the flow produced by a turbulent radial wall jet over a plane wall, at constant pressure, temperature and density. The momentum equation expressed in boundary-layer form is

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(\epsilon\frac{\partial u}{\partial y}\right), \tag{1}$$

where x and y are distances measured along and perpendicular to the wall, x being measured from the axis of symmetry. ϵ is the eddy-viscosity and u and v are the components of the mean velocity along x and y respectively. The continuity equation is

$$\frac{\partial}{\partial x}(xu)+\frac{\partial}{\partial y}(xv)=0, \tag{2}$$

which can be satisfied by introducing a Stokes stream function ψ defined by

$$xu=\frac{\partial\psi}{\partial y},\quad xv=-\frac{\partial\psi}{\partial x}. \tag{3}$$

A second fluid of the same density as the impinging fluid is introduced at the wall at the point $x=0$. We now assume that the rate of diffusion in the x direction is small compared with that in the y direction, analogous with the boundary-layer form of the momentum equation, and hence the diffusion equation becomes

$$\frac{\partial}{\partial x}(cux)+\frac{\partial}{\partial y}(cvx)=x\frac{\partial}{\partial y}\left(\epsilon'\frac{\partial c}{\partial y}\right), \tag{4}$$

where ϵ' is the diffusion coefficient and c is the mass concentration of the secondary fluid. Substituting (2) into (4) we get

$$u\frac{\partial c}{\partial x}+v\frac{\partial c}{\partial y}=\frac{\partial}{\partial y}\left(\epsilon'\frac{\partial c}{\partial y}\right). \tag{5}$$

The relationship between the eddy viscosity and the diffusion coefficient is defined by $N_s=\epsilon/\epsilon'$ where N_s is the Schmidt number.

If we consider the mass of secondary fluid introduced into and leaving a cylinder of radius x about the jet axis per unit time, we have in the steady state

$$2\pi\int_0^\infty xuc\,dy=\text{constant}=Qc_{00}, \tag{6}$$

where Q is the volume of secondary fluid introduced per unit time and c_{00} is its concentration.

The boundary conditions are

$$u\to 0,\quad c\to 0\quad\text{as}\quad y\to\infty;\quad u=v=0,\quad \partial c/\partial y=0\quad\text{at}\quad y=0, \tag{7}$$

since we are considering a single source in an otherwise impervious wall.

2.2. *Similarity solutions*

Non-dimensional variables are now introduced by writing

$$u = U\bar{u}, \quad v = U\bar{v}, \quad x = \frac{\nu\bar{x}}{U}, \quad y = \frac{\nu\bar{y}}{U}, \qquad (8)$$

$$\psi = \frac{\nu^2\bar{\psi}}{U} \quad \text{and} \quad c = c_{00}\bar{c},$$

where U is a constant velocity and ν is the kinematic viscosity.

Glauert (1956) shows that if a similarity solution of the momentum and continuity equation exists, with the maximum velocity $U_m \propto x^a$ and the jet width $\delta \propto x^b$, then for a turbulent radial wall jet a and b are related to a parameter α by

$$a = \frac{-9\alpha}{5+4\alpha}, \quad b = \frac{4+5\alpha}{5+4\alpha}. \qquad (9)$$

Glauert also shows that

$$\bar{\psi} = \bar{x}^{5-4b} f(\eta), \quad \eta = \frac{5-4b}{\lambda}\bar{y}\bar{x}^{-b} \qquad (10)$$

and

$$\epsilon = A\lambda\bar{x}^{3-3b}f'^{6}\nu, \qquad (11)$$

where λ and A are constants.

In solving (5) we look for a similarity solution for the concentration in the form

$$\bar{c} = \bar{x}^m G(\eta), \qquad (12)$$

where $G(\eta)$ and m are to be determined.

With these similarities it can be seen from (6) that

$$m = -(a+b+1),$$

i.e.

$$m = -\frac{9}{5+4\alpha}. \qquad (13)$$

From (3) and (10) we now have

$$u = \frac{5-4b}{\lambda} U\bar{x}^{4-5b} f' \qquad (14)$$

and

$$v = -U(5-4b)\left\{\bar{x}^{3-4b}f(\eta) - \frac{b}{\lambda}\bar{x}^{3-5b}\bar{y}f'(\eta)\right\}. \qquad (15)$$

From (8) and (12) we have

$$\frac{\partial c}{\partial x} = c_{00}\frac{U}{\nu}\bar{x}^{m-1}\left\{mG - \frac{b(5-4b)}{\lambda}G'\bar{y}\bar{x}^{-b}\right\} \qquad (16)$$

and

$$\frac{\partial c}{\partial y} = c_{00}\frac{U}{\nu}\frac{5-4b}{\lambda}\bar{x}^{m-b}G'. \qquad (17)$$

(11) and (17) now give

$$\frac{\partial}{\partial y}\left(\epsilon'\frac{\partial c}{\partial y}\right) = \frac{c_{00}A}{N_s\lambda}(5-4b)^2\frac{U^2}{\nu}\bar{x}^{m+3-5b}\frac{d}{d\eta}(f'^{6}G') \qquad (18)$$

and (5) becomes
$$\frac{d}{d\eta}\left(\frac{A}{N_s}f'^6G'\right) = \frac{m}{5-4b}Gf' - G'f. \tag{19}$$

In the outer part of the flow the same forms of the variables are retained except for $\epsilon(11)$, which is now considered to be constant through the layer,

$$\epsilon = \lambda\bar{x}^{3-3b}\nu. \tag{20}$$

Equation (5) now gives
$$\frac{G''}{N_s} = \frac{m}{5-4b}Gf' - G'f. \tag{21}$$

3. Integration

It is noted, from (9) and (13), that

$$\frac{m}{5-4b} = -1 \tag{22}$$

and hence that (19) and (21) reduce to

$$\frac{d}{d\eta}\left(\frac{A}{N_s}f_1'^6G_1'\right) + G_1f_1' + G_1'f_1 = 0 \tag{23}$$

and
$$\frac{G_0''}{N_s} + G_0f_0' + G_0'f_0 = 0, \tag{24}$$

where the suffixes refer to the inner and outer parts of the flow respectively.

From a similar treatment of the momentum equation (1) we obtain for the inner and outer regions respectively,

$$\frac{d}{d\eta}(Af_1'^6f_1'') + f_1f_1'' + \alpha f_1'^2 = 0 \tag{25}$$

and
$$f_0''' + f_0f_0'' + \alpha f_0'^2 = 0 \tag{26}$$

with boundary conditions

$$f_1'\eta^{-\frac{1}{7}} \to \text{constant} \quad \text{as} \quad \eta \to 0 \quad \text{and} \quad f_0'(\infty) = 0.$$

The junction of the inner and outer layer velocity profiles is taken to be the velocity maximum, where $\eta = \eta_m$; hence the concentration profiles are joined at the same point, where

$$\left.\begin{aligned} f_0''(\eta_m) &= f_1''(\eta_m) = 0, \\ f_0'(\eta_m) &= f_1'(\eta_m), \\ f_0(\eta_m) &= f_1(\eta_m), \\ G_0(\eta_m) &= G_1(\eta_m). \end{aligned}\right\} \tag{27}$$

Putting $f_2(\eta) = A^{\frac{1}{5}}f_1(\eta)$ into (23) and (24) we get

$$\frac{d}{d\eta}\left(\frac{f_2'^6G_1'}{N_s}\right) + f_2'G_1 + f_2G_1' = 0, \tag{28}$$

$$\frac{d}{d\eta}(f_2'^6f_2'') + f_2f_2'' + \alpha f_2'^2 = 0. \tag{29}$$

Glauert shows, by considering the relative importance of the second and third terms, that (29) can be replaced by

$$\frac{d}{d\eta}(f_3'^6 f_3'') + f_3 f_3'' + 2f_3'^2 = 0, \tag{30}$$

where
$$\left.\begin{aligned} f_3(\eta) &= Bf_2(\eta) \\ \text{and} \quad B^5 &= \frac{2{\cdot}07}{\alpha + 0{\cdot}07}. \end{aligned}\right\} \tag{31}$$

(30) can now be integrated to give

$$g'^7 = 1 - g^{\frac{9}{7}}, \tag{32}$$

where
$$g(\xi) = f_3^{\frac{7}{8}}(\eta) \quad \text{and} \quad \xi = \tfrac{7}{8}(\tfrac{56}{9})^{\frac{1}{7}}\eta. \tag{33}$$

Putting (31) into (28) and integrating with respect to η gives

$$\frac{f_3'^6 G_1'}{B^5 N_s} + f_3 G_1 = \text{constant} = 0 \tag{34}$$

from the boundary conditions $f_3'(0) = 0$ and $f_3(0) = 0$.

From (33) we have
$$f_3'^6 = (\tfrac{56}{9})^{\frac{6}{7}} g'^6 g^{\frac{6}{7}} \tag{35}$$

and
$$G_1' = \frac{dG_1}{d\xi}\tfrac{7}{8}(\tfrac{56}{9})^{\frac{1}{7}}. \tag{36}$$

Substituting (35) and (36) into (34) we have, using (32),

$$\frac{dG_1}{G_1} = -\tfrac{9}{49} B^5 N_s \frac{g^{\frac{2}{7}}}{(1-g^{\frac{9}{7}})}\,dg,$$

i.e.
$$\frac{G_1(\eta)}{G_1(0)} = (1-g^{\frac{9}{7}})^{B^5N_s/7},$$

$$\frac{G_1(\eta)}{G_1(0)} = \{1-(Bf_2)^{\frac{9}{8}}\}^{B^5N_s/7}. \tag{37}$$

Consider now the outer part of the layer. Integrating (24) with respect to η gives

$$\frac{G_0'}{N_s} + G_0 f_0 = \text{constant} = 0 \tag{38}$$

from the boundary conditions $G_0'(\infty) = 0$ and $G_0(\infty) = 0$. (38) can be rearranged in the form

$$\frac{dG_0}{G_0} = -N_s f_0\,d\eta. \tag{39}$$

Integrating (39) gives

$$G_0(\eta) = G_0(\eta_m)\exp\left[-N_s\int_{\eta_m}^{\eta} f_0(\eta)\,d\eta\right]. \tag{40}$$

In order to find $f_0(\eta)$, a solution of (26) must be sought. Glauert does this by assuming a series solution valid for large values of η and extending it to smaller values of η by numerical integration. Hence the integral of (40) can be evaluated. The concentration profiles are now joined at $\eta = \eta_m$ (η_m can be taken with sufficient accuracy from figure 3 of Glauert's paper for any value of α, α being found from the velocity profiles). Then from (37) we can find $G_1(\eta)$ at this point.

$$G_1(\eta_m) = G_1(0)\{1-[Bf_2(\eta_m)]^{\frac{9}{8}}\}^{B^5N_s/7}.$$

Then as

$$G_1(\eta_m) = G_0(\eta_m)$$

we have from (40)

$$\frac{G_0(\eta)}{G_1(0)} = \{1-[Bf_2(\eta_m)]^{\frac{9}{8}}\}^{B^5N_s/7}\exp\left[-N_s\int_{\eta_m}^{\eta} f_0(\eta)\,d\eta\right] \tag{41}$$

and hence the concentration profile across the whole of the jet can be found from (37) and (41) in terms of the concentration at the wall.

From the constancy of mass flow we can write, by dividing the region into its inner and outer flows,

$$\int_0^{y_m} xu_1c_1\,dy + \int_{y_m}^{\infty} xu_0c_0\,dy = \frac{Qc_{00}}{2\pi}. \tag{42}$$

The first integral gives

$$\frac{c_{00}\nu^2}{U}\bar{x}^{m+5-4b}G(0)A^{-\frac{1}{5}}\int_0^{f2m}\{1-[Bf_2(\eta)]^{\frac{9}{8}}\}^{B^5N_s/7}\,df_2$$

and the second one

$$\frac{c_{00}\nu^2}{U}\bar{x}^{m+5-4b}G(0)\{1-[Bf_2(\eta_m)]^{\frac{9}{8}}\}^{B^5N_s7}\int_{\eta_m}^{\infty} f_0'(\eta)\exp\left[-N_s\int_{\eta_m}^{\eta} f_0(\eta)\,d\eta\right]d\eta.$$

Denoting

$$A^{-\frac{1}{5}}\int_0^{f_{2m}}\{1-[Bf_2(\eta)]^{\frac{9}{8}}\}^{B^5N_s/7}\,df_2 + \{1-[Bf_2(\eta_m)]^{\frac{9}{8}}\}^{B^5N_s/7}\int_{\eta_m}^{\infty} f_0'(\eta)\exp\left[-N_s\int_{\eta_m}^{\eta} f_0\,d\eta\right]d\eta$$

by Z we have

$$c(x,0) = \frac{Qc_{00}U^{4b-4}x^{4b-5}}{2\pi Z\nu^{4b-3}} \sim x^{4b-5}. \tag{43}$$

The value of A can be found by considering the continuity of the velocity and the Stokes's stream function at the velocity maximum.

From (37), (41) and (43) the concentration at any point in the wall-jet region can be found.

4. Results

Figure 2 shows a plot of c/c_0 against η made from equations (37) and (41) for various values of N_s and with $\alpha = 1{\cdot}3$ where c_0 is the concentration at the wall. The velocity profile $u/u_{\max}$, where $u_{\max}$ is the velocity maximum, against η is also shown for $\alpha = 1{\cdot}3$. Although the gradient of the profile at the wall is zero, the wall having been assumed impervious, the curvature in this region is so great as to make this indiscernable. It is interesting to note that this feature was found both analytically and experimentally by Seban & Back (1961) in their investigation of temperature profiles in a two-dimensional tangential wall jet on an adiabatic wall. For $\alpha = 1{\cdot}3$ we can see from (43) that $c(x, 0)$ varies as $(x^{-0.88})$ which agrees closely with the corresponding variation of maximum temperature with distance in a heated jet $(x^{-0{\cdot}9})$ as found by Poreh & Tsuei (1965). However, as the theoretical variation of layer thickness with x has been found to differ slightly from experimental results, e.g. Bakke (1957), it is possible that some variation will also be evident here too.

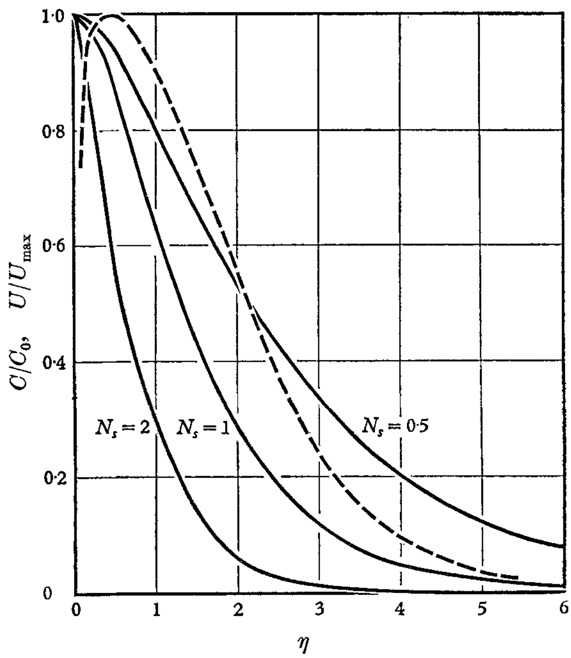

FIGURE 2. Concentration profiles at different Schmidt numbers and the velocity profile for $\alpha = 1{\cdot}3$.

REFERENCES

BAKKE, P. 1957 *J. Fluid Mech.* **2**, 467.
BLASIUS, H. 1913 *Forshungsarbeiten des Ver. deutsch Ing.* no. 131.
BRADSHAW, P. & GEE, M. T. 1962 *A.R.C.R & M* no. 3252.
BRADSHAW, P. & LOVE, E. 1961 *A.R.C.R. & M* no. 3205.
CHAUDHURY, H. 1964 *Mathematika*, **11**, 19.
FÖRTHMANN, E. 1934 *Ing.-Arch.* **5**, 42; also *N.A.C.A. Tech. Memo.* no. 789.
GLAUERT, M. B. 1956 *J. Fluid Mech.* **1**, 625.
POREH, M. & TSUEI, Y. G. 1965 *Israel J. Tech.* **3**, 7–10.
SCHLICHTING, H. 1955 *Boundary Layer Theory*. London: Pergamon Press.
SEBAN, R. A. & BACK, L. H. 1961 *Int. J. Heat Mass Transfer*, **3**, 255.
SIGALLA, A. 1958 *J. Roy. Aero. Soc.* **62**, 873.
TSUEI, Y. G. 1962 Ph.D. Thesis Colorado State University.